

Hans Grauert (1930-2011) *

Alan Huckleberry

March 28, 2013

Hans Grauert passed away at the age of 81 in September of 2011. His contributions to mathematics have and will be used with great frequency, and in particular for this reason will not be forgotten. All of us in mathematics stand on the shoulders of giants. For those of us who work in and around the area of *complex geometry* one of the greatest giants of the second half of the 20th century is Hans Grauert.

Specialists in the area know this, but even for them his collected works, annotated with the much appreciated help of Yum-Tong Siu, should at least be kept on the bedside table. An eloquent firsthand account of the *Sturm und Drang* period in Münster can be found in Remmert's talk (an English translation appears in [R]) on the occasion of Grauert receiving the von Staudt Preis in Erlangen. More recently, on the occasion of his receiving the Cantor Medallion, we presented a sketch of the man and his mathematics (see [H1, H2]). In the AMS-memorial article [AMS] specialists in the area, some of whom were students of Grauert, give us a closer look. In the present article we attempt to give an in-depth view, written for non-specialists, of Grauert's life in mathematics and the remarkable mathematics he contributed.

Early surroundings

Grauert was born in 1930 in Haren, a small town near the Netherlands in the northwestern part of Germany. Many of our friends who lived as children in this region recall their wartime fears, in particular of the bombings. Münster, which, together with the neighboring city Osnabrück, was the city of the

*AMS Subject classification 32C15, 32C55, 32E40, 32F10, 32J25, 32L99, 32Q15, 32Q15,01A61

signing of the Treaty of Westfalia ending the Thirty Years' War, was to a very large extent flattened. We never heard Grauert mention any of this; instead, he often told stories about having fun playing with unspent shells after the war, something that took the sight of another great complex analyst of the same generation, Anatole Vitushkin, far away in the Soviet Union. In [H1, H2] we recalled Grauert's detailed remarks about these days at his retirement dinner. In particular, he wanted to explicitly thank one of his grade school teachers for not failing him for his lack of skill in computing with numbers, informing him that soon he would be thinking in symbols and more abstractly.

Immediately after completing elementary school and his gymnasium education in nearby Meppen, Grauert began his studies in Sommersemester 1949 at the university in Mainz. In the fall of that year he transferred back to Münster, where he would go from schoolboy to one of the worldwide leading authorities in the area of several complex variables and holder of the Gauss Chair in Göttingen in a period of ten years.

Despite the destruction caused by the war (Germany was only beginning to *rise from the ashes*), Münster was one of the best places in the world to start out in complex analysis. At the leadership level Heinrich Behnke, Henri Cartan and Karl Stein were playing key roles. Among the students there were already the likes of Friedrich Hirzebruch, who began his studies in 1945, and Reinhold Remmert, who would become Grauert's lifelong friend and co-author of numerous fundamental research articles and expository monographs.

Behnke had come to Münster in 1927 as a proven specialist in the complex analysis in several variables of the time. Fundamental first results had already been discovered and proved. These include the remarkable facts about the location and nature of singularities of holomorphic functions proved by two giants of the early 20th Century, Eugenio Elie Levi and Friedrich Hartogs. To give a flavor of the times, let us summarize a bit of this mathematics.

Levi had understood that if the smooth boundary of a domain D in \mathbb{C}^n , $n \geq 2$, is locally defined as $\{\rho = 0\}$ with ρ being negative in the domain, then the complex Hessian $(\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j})$ contains curvature information which determines whether or not a holomorphic function on the domain can be continued holomorphically across the boundary point in question. For example,

the appropriate notion of concavity at a boundary point p with respect to this *Levi form* can be restated by requiring the existence of a holomorphic mapping F from the unit disk Δ in the complex plane to the closure of D with $F(0) = p$ and $F(\Delta \setminus \{p\}) \subset D$. If ∂D is concave at p , then every function holomorphic on D extends holomorphically to a larger domain \hat{D} which contains p in its interior.

Hartogs had understood related phenomena, the simplest example of which goes as follows. Consider a domain D , e.g., in \mathbb{C}^2 which can be viewed as a fiber space by the projection onto the unit disk Δ in the first variable. For some arbitrarily small neighborhood Δ' of the origin in Δ the fibers are assumed to be unit disks in the space of the second variable and otherwise are annuli with outer radius 1 and inner radius arbitrarily near 1. Then every holomorphic function on D extends holomorphically to the full polydisk $\hat{D} = \{(z_1, z_2) : |z_i| < 1, i = 1, 2\}$.

The so-called Cousin problems, formulated by Cousin in a special context in the late 19th century, which when positively answered are the higher-dimensional analogues of the Mittag-Leffler and Weierstrass product theorems of one complex variable, are also related to questions of analytic continuation. For example, Cousin I for a domain D in \mathbb{C}^n , asks for the existence of a globally defined meromorphic function on D with (locally) prescribed principal parts. This means that on an open covering $\{U_\alpha\}$ of D there are given meromorphic functions m_α which are compatible in the sense that $m_\beta - m_\alpha =: f_{\alpha\beta}$ is holomorphic on the intersection $U_{\alpha\beta}$. The question is whether or not there is a globally defined meromorphic function m on D with $m - m_\alpha$ holomorphic on U_α for every α .

The following is a connection of Cousin I to the study of analytic continuation. Let D be a domain in \mathbb{C}^n with $p = 0$ in its smooth boundary and suppose that the set $\{z_1 = 0\}$ is locally in the complement of D . In order to show that some holomorphic function on D cannot be continued across p , one could try the following: Bump out D at 0 to obtain a slightly larger domain \hat{D} which contains an open neighborhood U_0 of p . Define $U_1 = D$ and consider the Cousin I data for the covering $\{U_0, U_1\}$ of \hat{D} of $m_1 \equiv 0$ and $m_0 = \frac{1}{z_1}$. If this has a “solution” m , then $m|_D$ is an example of a holomorphic function that cannot be continued through p ! We mention this here, because, as we explain below, Grauert brought this Ansatz to fruition and perfected it in its ultimate beauty.

Behnke certainly knew that several complex variables was an area ripe for development and set about building a research group for doing so. He was an active mathematician who understood where mathematics was going and where it should go. He was optimally connected to the world outside Münster. Caratheodory, Hopf, Severi and many others were his close friends. Perhaps above all, he was a remarkable organizer of all sides of our science! It must be emphasized, however, that he was fortunate! Even early on he had a group of magnificent students/assistants, three of whom we have had the honor of knowing: Peter Thullen, Friedrich Sommer and Karl Stein. Secondly, for the seemingly innocuous reason that he had proved a small remark on circular domains which improved on an old result of Behnke, in 1931 Henri Cartan was invited by Behnke to give a few talks in Münster.

Thullen and Cartan became good friends and proved a basic result characterizing *domains of holomorphy* which possess holomorphic functions which cannot be continued across their boundaries. Thullen went on to prove a number of results, including important continuation theorems. Behnke and Thullen published their *Ergebnisbericht* which in particular outlines the key open problems of the time. In a series of papers which are essential for certain of Grauert's works, Kyoshi Oka solved many of these problems. Story has it that he wrote Behnke and Thullen a thank-you note for posing such interesting questions.

Friedrich Sommer, who was one of the founding fathers of the Ruhr University and who was responsible for continuing the tradition of complex analysis in Bochum, was one of the stalwarts of the Behnke group which Grauert joined.

Before the war, Behnke was still active in mathematics research, in particular with Stein who after the war became the mathematics guru of complex analysis in Münster. (See [H3] for a detailed discussion of Stein's contributions.) The work of Behnke and Stein underlining approximation theorems of Runge type, and, for example, Stein's emphasis on implementing concepts from algebraic topology (he spent time in Heidelberg with Seifert) certainly influenced the young Grauert.

Not enough can be said about the importance of Henri Cartan for the *Münsteraner school of complex analysis*. The pre-war interaction indicated above was just the beginning. Despite the fact that the Nazi atrocities directly touched Cartan's family (his brother was assassinated in 1943), shortly

after the war, in 1947, he accepted Behnke's invitation to visit Münster. For those of us who did not experience the horrific events of that time, it is difficult to imagine the magnitude of importance, maybe most importantly at the human level, of Cartan's reestablishing the Paris–Münster connection (see [HP] for more on the importance of Cartan for postwar German mathematics). The importance for complex analysis, in particular for Hans Grauert, is discussed below.

Initial conditions

When Grauert arrived in Münster, despite the fact that the worldly amenities of the university were still at best minimal, Behnke had complex analysis up and running and, in a certain sense, the conditions for research were optimal. On the one hand there was Stein, a kind, modest man of enormous enthusiasm and energy who had deep insight at the foundational level of, e.g., analytic sets, holomorphic mappings, etc. Certainly the Cousin problems and their relationships to domains of holomorphy had guided a big part of his thinking. As a result of research with Behnke before the war and published in 1948, he knew these were solvable on non-compact Riemann surfaces. To add a bit to a paper which he worried was otherwise too short he formulated three axioms for what are now known as *Stein manifolds* which he felt would be the correct general context for solving problems of Cousin type ([S]): Globally defined holomorphic functions separate points and give local coordinates, and given a divergent sequence $\{z_n\}$ there exists a holomorphic function f with $|f(z_n)|$ unbounded.

Stein was a hands-on craftsman and this certainly influenced the spirit of the Behnke seminar where there were lengthy naïve (healthy!) discussions of examples such as \sqrt{xy} (the cone singularity $z^2 - xy = 0$ which can be viewed as a 2:1 cover of \mathbb{C}^2 ramified only at the origin). On the other hand, the mathematics world outside Münster, particularly influenced by developments in France, had made a quantum leap in sophistication. However, Behnke made sure that Münster was not isolated.

Fritz Hirzebruch had begun his studies in Münster in 1945. He lived in Hamm where Stein also lived. We have heard that they traveled to Münster by train together often hanging on to the outer running boards with Stein making propaganda for the role of algebraic topology in complex geometry.

During Hirzebruch's studies, Behnke sent him to his friend Hopf in Zurich. Hirzebruch happily reminisced about learning from Hopf about blowing up points and blowing down curves in surfaces. In fact his thesis (published in 1951) is a jewel about surfaces where this process plays a role. On another not unrelated topic, in a talk to a historical society on the Riemann-Roch theorem, Hirzebruch said that probably the most important new development for him in the early 1950's was understanding of the notion of a line bundle! Just a few years later he fused a hefty portion of the new sophistication with his own ideas to prove his Riemann-Roch Theorem!! (published in 1954.) For the young Münsteraner it must have been extremely motivating to see this remarkable development.

Cartan's early works, e.g., with Thullen and those on automorphism groups of domains, fit in the style of complex analysis at the time. However, Cartan not only made the leap from the classical to the post-war level of sophistication, he was one of the main figures who shaped it. Despite having formulated and proved Theorems A and B (published in 1951), which not only solve the Cousin problems on Stein spaces but put complex analysis in another world of abstraction (the distance from Münster to Paris could no longer be measured in kilometers), he remained in contact with and supported the members of the Behnke group. By the way, it was his idea to refer to these spaces as *Variété de Stein*.

Crescendo

At this point in the historical timeline Grauert entered the picture and, at certain points with the help of distinguished co-workers, took complex analysis to yet another level. Having set the stage above, we now turn to a description of representative aspects of his published works. We begin with an overview.

Grauert received his doctorate in Münster in mid-1954. His first publications appeared in 1955, the publication from his thesis in 1956. In the five or six years that followed, his contributions to mathematics were truly remarkable: a wealth of ideas, numerous basic results and simply quite a number of published pages. Disregarding research announcements (Comptes Rendus Notes), conference reports and expository articles, in this intense period he authored or co-authored (with Remmert, Andreotti and one with his student Docquier) 19 articles which covered a total of roughly 600 pages. In

the three or four years after this period, when both he and Remmert were in Göttingen, they jointly wrote three basic research monographs in book form: *Analytische Stellenalgebren*, *Stein Theory* and *Coherent Analytic Sheaves*. The final versions of the latter two appeared much later. After settling in Göttingen, where he also devoted a great deal of time to his students (he guided more than 40 Ph.D. theses), Grauert continued to make important research contributions. Altogether he published more than 90 works, most of which were devoted to topics in the areas of several complex variables and complex algebraic geometry.

In a nutshell one can summarize Grauert's work as being fundamental for the foundations of the geometric side of complex analysis, particularly his early work with Remmert, and for our understanding of the multifaceted global phenomena related to Levi curvature. His solution of a certain Levi problem is just one of a number of results in this direction. There are two early works of Grauert that stand out as the peaks among many mountains: The Oka Principle (1957) and the Direct Image Theorem (1960). These and selected works in the areas indicated above are discussed in some detail in the next section.

In addition to those works which will be discussed in the next section, a number of important papers must be mentioned, e.g., that on the solution of the Mordell conjecture in the function field case. Weil mused that Manin, the algebraist, used analytic methods for this whereas Grauert, the analyst, approached it algebraically. In fact, if one looks at the paper, one immediately sees Grauert's geometric viewpoint. Other results which stand out are his construction of the versal deformation space for compact complex spaces (simultaneously with Douady) and that for deformations of isolated singularities, his basic cohomology vanishing theorem with Riemenschneider and results on conditions for the formal equivalence of neighborhoods of analytic subsets implying convergent equivalence. His work with Müllich on vector bundles on \mathbb{P}_2 has been extremely influential. Fundamental work on the analytic side, in particular solving $\bar{\partial}$ -problems with bounded data, was carried out with his students, Ramirez and Lieb. He also wrote textbooks for basic real analysis with Lieb and for linear algebra with Grunau. In the area of several complex variables he wrote two textbooks with Fritzsche, together with Peternell and Remmert he edited and contributed several chapters to a volume of the Encyclopedia of Mathematical Sciences and wrote the three

research monographs with Remmert which were mentioned above.

Grauert considered a wide range of topics. For example, one should not forget his ideas on hyperbolicity (see Demailly's comments in [AMS]) as well as his interests in vector bundles, deformation theory and in understanding analytic equivalence relations, a topic that had followed him since his early encounters with Karl Stein. He had a philosophical side as well which went along with his desire to understand certain kinds of physical (quantum mechanical) phenomena. It seems that he read Riemann's work having this in mind and, based on this, developed his own theory of discrete geometry. We recall his series of lectures at Notre Dame on his axiomatic approach and note that at the end of Volume II of his collected works he included several pages on this. Given that he obviously carefully polished these two volumes, it is clear that he took this subject very seriously and that it meant a great deal to him.

Comments on selected works

Under the given time and space constraints it is only possible to present a small sample of Grauert's works. Since he is perhaps best known for his results at the foundational level in complex geometry, those involving Levi geometry, his Oka principle and his proof of the direct image theorem, our remarks here will focus on these subjects.

Early days

We begin with comments on certain aspects of Grauert's dissertation (published in [Gr2]) which underlined an important connection between complex differential geometry and complex analysis. This is followed by remarks on the paper where he vastly improved our foundational understanding of Stein spaces ([Gr1]). This early work emphasized the need for building the foundations of *complex spaces*. Much in this direction was accomplished in the basic paper *Komplexe Räume* ([GR1]) of Grauert and Remmert which is the third paper we review in this paragraph.

Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik

When Grauert went to the ETH (1953), it was already quite fashionable to study Kähler manifolds. By definition such a manifold possesses a Hermitian

metric whose imaginary part is a closed 2-form. Locally this form ω has a potential φ , i.e., $\omega = \frac{i}{2}\partial\bar{\partial}\varphi$, and the positivity of the metric translates to φ being strictly plurisubharmonic. Hodge's book, which appeared in 1941, was well known and Eckmann and Guggenheimer were busy in Zurich (Guggenheimer went to Israel in 1954) looking at more general manifolds. The fact that plurisubharmonic functions (Lelong, Oka 1942) are important in complex analysis was widely understood. Kähler himself had thought in terms of the potential function and had in fact proved that Kähler is, as mentioned above, the same thing as having a locally defined strictly plurisubharmonic potential. Much was in the air when Grauert started thinking in this direction.

At the beginning of his paper Grauert states that it is “naheliegend” to study the connection between complete Kähler metrics and domains of holomorphy D . In hindsight this is true, but he was the first to do this. For our purposes a *domain of holomorphy* is a domain in \mathbb{C}^n for which there exists a function f holomorphic on D which cannot be extended to a function holomorphic on a larger manifold. In fact, given a sequence $\{z_n\}$ which converges to every boundary point, one can construct f with the property that $\lim |f(z_n)| = \infty$.

Grauert begins the article by pointing out that, given a complete Kähler metric on D and a closed analytic subset A of D , i.e., a set defined as the common 0-set of finitely many holomorphic functions, it is a simple matter to adjust the given Kähler metric appropriately to obtain a complete Kähler metric on $D \setminus A$. Since holomorphic functions extend across analytic sets which have codimension at least two, it is then immediate that there are domains with complete Kähler metrics which are not domains of holomorphy.

After making the above remark, Grauert then proves that the desired result holds if D has a \mathbb{R} -analytic boundary. In other words, such domains are domains of holomorphy if and only if they possess a complete Kähler metric. This result, which required substantial technical work, stimulated a great deal of further research, first of all due to the idea that the existence of a complete Kähler metric and pseudoconvexity are related. The regularity question also turned out to be of interest. For example, some years later Ohsawa showed that only a C^1 -boundary is necessary ([O]).

This paper is Grauert's doctoral thesis. He profusely thanks Behnke and Eckmann. I would guess that Eckmann and Guggenheimer discussed Kählerian

geometry with him, although their work exclusively dealt with compact manifolds and in particular had nothing to do with pseudoconvexity. Grauert received his degree in Münster in July of 1955, the first referee being Behnke and the second Friedrich Sommer.

Characterisierung der holomorph vollständigen komplexen Räume

In this paper Grauert is clearly fascinated by the question of countability of the topology of complex spaces (Rado noticed this in the case of Riemann surfaces). Throughout the paper a complex space is an α -space and Grauert is thinking in terms of it locally being the graph of a multivalued holomorphic function. He does not yet have the result that every α -space is a β_n space (see our review of *Komplexe Räume* for the notation). As a result he proves his main results under assumption C which due to the later work of Grauert and Remmert just means that a complex space is locally the common 0-set of finitely many holomorphic functions on a domain in \mathbb{C}^n .

The *new* axiom here is that of K -Vollständigkeit, i.e., that global holomorphic functions define a map at a given point which is finite fibered near the point in question. Much work then shows that under this condition the topology is countable and finally that the n -dimensional space X is globally a ramified Riemann domain over \mathbb{C}^n , i.e., that there is a generically maximal rank holomorphic map $F : X \rightarrow \mathbb{C}^n$. Then, using nontrivial (but more or less classical) methods, Grauert shows that if X is holomorphically convex, then it is Stein. The main result then is an essential weakening of Stein's axioms: K -vollständig plus holomorphic convexity are equivalent to the following four axioms of Stein:

1. Countable topology.
2. Globally defined holomorphic functions give local embeddings.
3. Globally defined holomorphic functions separate points.
4. Holomorphic convexity, i.e., given a divergent sequence $\{x_n\}$ there exists a holomorphic function f with $\lim |f(z_n)| = \infty$.

Komplexe Räume

Here the authors work from the point of view of the definition of Behnke and Stein which is that a complex space X is locally the graph of a multivalued

holomorphic function on \mathbb{C}^n . This was nothing new in the 1-dimensional case, because under the assumptions of Behnke and Stein the resulting space is smooth and locally just the graph of an algebraic function! However, in the higher dimensional case singularities arise. Members of the Behnke seminar in those years have told us that they spent great energy trying to understand $\pm\sqrt{xy}$ which is the cone defined by $z^2 - xy = 0$ over the xy -plane!

To be precise, a Behnke-Stein complex space is a Hausdorff space X which satisfies the following local condition: Every $x \in X$ is contained in an open neighborhood U which is equipped with a continuous map $\varphi : U \rightarrow V$ onto an open set in \mathbb{C}^n which contains a proper analytic subset A with the property that the restriction $\varphi : U \setminus \varphi^{-1}(A) \rightarrow V \setminus A$ is a proper finite covering map. In particular, φ is a local homeomorphism and gives local holomorphic coordinates on $U \setminus \varphi^{-1}(A)$. The holomorphic functions on U are then defined to be the continuous functions which are holomorphic on $U \setminus \varphi^{-1}(A)$. One should mention in this context that one of the most quoted theorems of Grauert and Remmert is that if Y is, for example, a complex manifold which contains a proper analytic subset A and $F : X \rightarrow Y \setminus A$ is a proper unramified holomorphic map, then X can be (uniquely) realized as the complement of a proper analytic subset B in a larger complex space \bar{X} so that the analytic cover $X \rightarrow Y \setminus A$ can be extended to a proper (finite) holomorphic map $\bar{X} \rightarrow Y$ with $\bar{X} \setminus B \rightarrow Y \setminus A$ being the original map.

The main goal of this paper is to show that the definition of Behnke and Stein is equivalent to that of Cartan and Serre of a *normal* complex space. To “clarify” matters Grauert and Remmert introduce a rather cumbersome notation. First, the Behnke-Stein spaces are called α -spaces. The spaces coming from Paris are called β - and β_n -spaces. The former is locally the common 0-set A of finitely many functions on a domain D in some \mathbb{C}^n (of course depending on the point) and the sheaf of holomorphic functions is just the quotient $\mathcal{O}_D/\mathcal{I}_A$ of the sheaf of germs of holomorphic functions on D by the full ideal sheaf of functions which vanish on A . In modern terminology these are just called reduced complex spaces. The β_n spaces are those which are *normal* in the sense that if a meromorphic germ satisfies a monic polynomial equation with holomorphic coefficients, then it is itself holomorphic. Due to applications it was and still is important to understand the relations among these concepts. The Grauert-Remmert paper clears this up completely. Furthermore several basic results of independent interest are

proved.

It is not terribly difficult to show that if the ramification on an α -space is given by a multivalued function (algebroid condition), then that space is a β_n -space. That, then, is the main theorem of the paper: α spaces are automatically algebroid. Since β_n -spaces are easily seen to be α -spaces, this now completes the circle: The Behnke-Stein spaces are exactly the normal complex spaces defined by Cartan!

Two basic results which we have not yet mentioned were proved along the way: 1.) A β -space is normal if and only if the Riemann extension theorem holds. 2.) The normalization of a β -space is constructed. By *Riemann extension* we mean that if A is a proper analytic subset of X and f is holomorphic on $X \setminus A$ and is locally bounded near A , then it extends to a holomorphic function on X . The normalization $\pi : \hat{X} \rightarrow X$ of a complex β -space, is a finite (proper, surjective) holomorphic map from a canonically determined normal complex space which is biholomorphic at least outside of the singular set of X . For $x \in X$ the number of points in $\pi^{-1}(x)$ is the number of local irreducible components of X at p . It is quite possible that the normalization is a homeomorphism with the only difference between \hat{X} being that the structure on \hat{X} is richer.

The reader should consult *Coherent Analytic Sheaves* [GR2] for a more modern formulation of the above. On the other hand, this book was written in note form in Göttingen in the early to mid-1960s, i.e., not long after the original article.

Oka principle

Grauert wrote three papers ([Gr3, Gr4, Gr5]) on what is now called Grauert's Oka principle. The first two are full of deep ideas and fundamental work. The third is devoted to a statement of what was proved in the first two along with several applications. After giving some background on what was known when Grauert entered the picture, with the help of Cartan's formulation and streamlining we outline the statements and ingredients of proof of Grauert's results. These can be seen as proving that on a Stein space the only obstructions to solving problems of a complex analytic nature are topological.

Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen

Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen

Analytische Faserungen über holomorph-vollständigen Räumen

The following is a simple but fundamental example of the Oka principle. Let G be a domain in \mathbb{C}^n and D be a divisor on G , i.e., D is given locally on a covering $\{U_i\}$ by meromorphic functions m_i which satisfy the compatibility condition that $m_i = f_{ij}m_j$ on the intersection $U_i \cap U_j =: U_{ij}$ with the f_{ij} being nowhere vanishing holomorphic functions on the intersections $U_{ij} = U_i \cap U_j$. One might ask (the second Cousin problem) if there is a globally defined meromorphic function m on D with this divisor. In other words, m would be required to have the same poles and zeros (counting multiplicity) as the m_i in the sense that the functions $\frac{m}{m_i} =: f_i$ are holomorphic and nowhere vanishing on the U_i . This is of course the same as asking for the existence of such f_i with $f_i m_i$ being globally defined.

On the complex plane, i.e., for $D = \mathbb{C}$, the question of existence of the global function is answered in the positive by describing one such as the quotient of Weierstrass products. As a consequence of the Theorem of Behnke and Stein this even holds for every non-compact Riemann surface. On the other hand for compact Riemann surfaces, and of course for higher-dimensional compact complex manifolds, such a theorem does not hold and the fact that it does not guides many questions in the theory.

Returning to the non-compact case, if $D = \mathbb{C}^n$, the second Cousin problem can also be answered in the positive, even by using methods that are analogous to the Weierstrass products. However, it was realized early on that without further assumptions, even for domains D in \mathbb{C}^n , there would be no hope of solving this problem. The appropriate class of domains which appeared natural for solving this problem is the class of Stein domains or domains of holomorphy. Such is the natural domain of existence of some holomorphic function f in the sense that it cannot be extended holomorphically to any larger complex manifold. Although Stein domains are optimal from many points of view, as was realized by Oka, even on Stein domains there are obstructions to solving such problems. The point is that there might not even exist continuous functions f_i with this property. In our more

modern language, the f_{ij} define a holomorphic line bundle $L(D)$ on D and the Cousin II problem has a positive solution if and only if $L(D)$ is holomorphically trivial. The problem has a continuous solution if and only if $L(D)$ is topologically trivial. Oka's basic theorem, proved in the pre-war years, states that a holomorphic line bundle on a Stein domain is holomorphically trivial if and only if it is topologically trivial. Cartan's Theorem B, or the more general theorem of Grauert (see above), immediately implies this statement on Stein spaces. It should be remarked that there are no topological obstructions to solving the additive (Cousin I) problem, i.e., that which asks for a globally defined meromorphic function with prescribed principle parts. On a Stein space it always has a positive solution.

For the reason sketched above, and for various other questions which arise in complex analysis, a vague Oka principle can be formulated: A problem on a Stein space which is formulated in complex analytic terms has a complex analytic solution if and only if it has a topological solution. This was more or less the state of the theory when Grauert entered the picture with his three papers which were published in 1957. Although it does no justice to his work, one simple-to-state consequence of Grauert's Oka principle is that on a Stein space the mapping which forgets complex structure defines an isomorphism between the categories of holomorphic and topological vector bundles. In other words, Oka's theorem holds for arbitrary rank.

Unlike the case of divisors where the transition matrices f_{ij} have values in an Abelian group and the usual cohomological technology on Stein spaces can be applied, in Grauert's non-Abelian setting classical results of the time cannot be applied. On the other hand it was certainly clear that in order to handle the Oka principle, e.g., some version of the classical Runge approximation theorem would be necessary. In fact an extremely deep version of this approximation theorem, one which involves homotopies of holomorphic and continuous maps, would be of essential importance in Grauert's theory.

Grauert opens his first paper as follows: *In the present paper functions $F(r)$ from a complex space R with values in an arbitrary complex space W will be studied.* Of course he has in mind Stein spaces; so let us assume that R is Stein and consider a Stein space \tilde{R} which contains R . The main question is if such a "function" F can be approximated (uniformly on compact subsets) by a holomorphic function from \tilde{R} to W . For usual functions with values in \mathbb{C}

the Behnke-Stein theorem was available: The approximation theorem holds if and only if R is *holomorph ausdehnbar* to \check{R} . This is a condition which is a bit complicated to formulate. It is a substantially weaker version of there being a continuous increasing family R_t , $0 \leq t \leq 1$, of Stein domains with $R_0 = R$ and $R_1 = \check{R}$. In any case, since this condition is already necessary and sufficient for usual functions, it was clear to Grauert that he should assume it for his more general question. This being clear, Grauert jumps to a suitable setting.

Following his notation, a Lie group bundle over a complex space R is a holomorphic fiber bundle $L^*(R, L)$ with fiber a complex Lie group L and structure group L^* contained in the group of Lie group automorphisms of L . Note that the case of $L = (\mathbb{C}^n, +)$ is that of a holomorphic vector bundle. Observe that such a bundle has the identity section, fiberwise multiplication makes sense, one has the associated Lie algebra bundle where the exponential is biholomorphic near its 0-section and cohomology concepts for the sheaf of sections make sense using the group multiplication. Note also that such a bundle is *not* a principal bundle.

Before going further we would like to simplify the notation and follow that of Cartan's paper ([C]) where he explains Grauert's work with great elegance. He simply denotes the Lie group bundle by $E \rightarrow X$ and then introduces the notion of an E -principal bundle defined by a cocycle $\{f_{ij}\}$ acting on L on the left. In this way one has the fiberwise action $F \times_X E \rightarrow F$ on the right. With this in mind one of Grauert's main theorems can be formulated as follows.

Let \mathcal{E}_c be the sheaf of continuous sections and \mathcal{E}_a be the sheaf of holomorphic sections of the Lie group bundle E .

Theorem. *The inclusion $\mathcal{E}_a \hookrightarrow \mathcal{E}_c$ induces an isomorphism*

$$H^1(\mathcal{E}_a) \cong H^1(\mathcal{E}_c).$$

Of course one of the main issues to be handled is that of a Runge theorem. Here is Grauert's Runge theorem for beginners.

Theorem. *Let R and \check{R} be Stein spaces with R holomorph ausdehnbar to \check{R} and let E be a Lie group bundle on \check{R} . Then a holomorphic section of E over R can be approximated by holomorphic sections of \check{R} if and only if it can be approximated by continuous sections on \check{R} .*

This should be regarded as a mini-Runge theorem, because the final version must be proved in a context where homotopy is involved. Let us leave this in Grauert's language.

Theorem. *Ist \mathfrak{R} ein holomorph-vollständiger Raum, so gibt es zu jeder in $\mathfrak{R} \times \mathfrak{T}_1$ definierten (e, h) -Funktion mit Werten in einem Faserraum $L^*(\mathfrak{R}, L)$ eine (e, h, c) -homotope (e, h^0) -Funktion.*

Here is Cartan's formulation.

Notation (The (N, H, K) -sheaf \mathcal{F}):

- K is an auxiliary compact parameter space with $N \subset H \subset K$ such that N is a deformation retract of K .
- $\mathcal{F}(U)$ is the topological group of continuous sections $s(x, t) : U \times K \rightarrow E(U)$ which are the identity section for $t \in N$ and holomorphic for $t \in H$.

Theorem. *If X is Stein, then*

1. $H^0(X, \mathcal{F})$ is arcwise connected.
2. If U is holomorphically convex in X , then image of the restriction $H^0(X, \mathcal{F}) \rightarrow H^0(U, \mathcal{F})$ is dense.
3. $H^1(X, \mathcal{F}) = 0$

Of course we have swept a great deal of work under the table. Implementing in particular the refined Runge theorem, Grauert spends a great deal of time solving the non-Abelian Cousin problems.

Here are some of the consequences of Grauert's remarkable work.

- Every continuous section $s : X \rightarrow F$ is homotopic to a holomorphic section.
- If two holomorphic sections are homotopic in the space of continuous sections, then they are homotopic in the space of holomorphic sections.
- Every continuous isomorphism between E -principal bundles F and G is homotopic to a holomorphic isomorphism.

Note that for the last result it is important that the quotient $E_g := (G \times_X G)/E$ is a Lie group bundle and the bundle of isomorphisms from G to F is the E_g -principal bundle $(F \times_X G)/E$.

There are numerous consequences of these results, and even recently there has been a big explosion of further developments (see [F]). A big additional step, both technically and conceptually, was Gromov's h -principal ([G]).

Levi convexity and concavity

Grauert made numerous fundamental contributions to understanding the role of Levi-curvature in complex analysis. If for example a domain D with smooth boundary in a complex manifold is defined by a smooth function, $D = \{\rho < 0\}$, it has been known since the beginning of the 20th century that signature invariants of the complex Hessian of ρ along ∂D play an essential role in determining the complex analytic nature of D . Grauert solved the version of the *Levi problem* ([Gr6]) which states that if the appropriate curvature form is positive-definite, i.e., D is strongly pseudoconvex, then D is essentially a Stein manifold. We review this basic paper here along with four other works where convexity, concavity or both are essential ingredients.

In his paper with Docquier ([DG]) it is shown that if D is contained in a Stein manifold and is weakly pseudoconvex (in any of a variety of ways) at the boundary, then it is Stein. (Oka and others had shown this for domains in \mathbb{C}^n .) Andreotti and Grauert wrote two very interesting papers where concavity is involved. One can be regarded as a mixed signature version of Grauert's previously handled positive-definite case where higher cohomology spaces replace function spaces ([AG1]). In the other they describe the structure of the field of meromorphic functions on a pseudoconcave space and show how to apply their methods to situations where the space at hand is a discrete group quotient, e.g., where the meromorphic functions arise as quotients of modular forms ([AG2]).

Finally, we review Grauert's beautiful paper *Über Modifikationen und exzeptionelle analytische Mengen* ([Gr8]). In brief, here he shows us how to use strong pseudoconvexity in the theory of compact complex spaces, in particular to settings of algebraic geometric interest. The title indicates one of the themes in the paper where he proves that a compact complex subvariety in a complex space can be blown down if and only if its normal bundle satisfies

a natural curvature condition. This is just one of many other results which are proved, e.g., new ampleness criteria, Kodaira type embedding theorem, etc..

On Levi's Problem and the Imbedding of Real-Analytic Manifolds

Let us begin here with a classical situation where D is a domain with smooth boundary in \mathbb{C}^n . Every point $p \in \partial D$ has an open neighborhood U which is equipped with a smooth function ρ with nowhere vanishing differential so that $U \cap D = \{\rho < 0\}$. In particular, $\partial D \cap U = M$ is the smooth hypersurface $\{\rho = 0\}$. Note that the full (real) tangent space $T_p M$ contains a unique maximal complex subspace $T_p^{CR} M$, the *Cauchy-Riemann tangent space* to M at p , which is 1-codimensional over \mathbb{R} . Let us say that the Levi-form $L_p(\rho)$ of the defining function ρ at p is the restriction to the CR-tangent space of the complex Hessian

$$Hess_p(\rho) = \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \right).$$

One can show that the signature of L is a biholomorphic invariant. If for example L is positive-definite, then supposing that $p = 0$ and that the CR-tangent space is given by $z_1 = 0$ one can introduce holomorphic coordinates (z_1, z') so that the restriction of ρ to $\{z_1 = 0\}$ is

$$\rho(z) = \|z'\|^2 + O(3).$$

Thus locally near p the CR-tangent space lies outside D and except at p is contained in the complement of the closure of D . One can think of it as a (local) supporting complex hypersurface outside D at the point p . If the Levi-form is positive-definite at every point of ∂D , one says that the domain (or its boundary) is strongly pseudoconvex. If at each point $p \in \partial D$ the Levi-form is only positive semidefinite, one just says that D is pseudoconvex. One can imagine that there is a huge difference between the concepts, particularly if the rank of L_p is allowed to vary wildly with the point p .

In the early part of the 20th century E. E. Levi realized that if L_p is not positive-semidefinite, then every function defined and holomorphic on D near p extends holomorphically across ∂D at p . On the other hand, if it is positive-definite, locally in the coordinates used for the above normal form, the function $\frac{1}{z_1}$ is holomorphic on D near p and does not continue across the boundary.

Therefore one asks if the same holds at the global level. Levi himself showed that if at some $p \in \partial D$ the Levi-form is not positive semidefinite, then every function holomorphic on D continues across ∂D . The Levi-Problem can be stated as follows:

$$D \text{ pseudoconvex} \stackrel{?}{\Rightarrow} D \text{ is a domain of holomorphy.}$$

In other words, if D is pseudoconvex, given a divergent sequence $\{z_n\}$ one would like to prove that there exists a holomorphic function f on D with $\lim |f(z_n)| = \infty$. Recall that this property is the only one of Stein's axioms that is not automatically fulfilled for a domain in \mathbb{C}^n . Hence, for domains one is really asking if pseudoconvex domains are Stein with the above property being called *holomorphic convexity*.

In 1942 Oka solved the problem for pseudoconvex domains unramified over \mathbb{C}^2 , and in 1953/54 this was extended to arbitrary dimensions independently by Oka, Bremermann and Norguet. Grauert points out that by using the Behnke-Stein theorem (limits of domains of holomorphy are domains of holomorphy) in \mathbb{C}^n , one needs only to prove the result for strongly pseudoconvex domains. This is not true for domains in arbitrary manifolds. For example, it is a simple matter to construct a domain D in a torus with Levi-flat boundary, i.e., the Levi-form of an appropriate boundary defining function vanishes identically (nevertheless D is pseudoconvex!), with the property that every holomorphic function on D is identically constant. Grauert has constructed much more sophisticated examples in [Gr9], where except for a small set ∂D is strongly pseudoconvex.

Grauert states his theorem for bounded domains D with smooth boundaries in arbitrary complex manifolds:

$$\text{strongly pseudoconvex} \Rightarrow \text{holomorphically convex.}$$

Actually he proves much more: *If D is strongly pseudoconvex, then it contains finitely many pairwise disjoint maximal compact analytic subvarieties which can be blown down so that the resulting complex space is Stein.* For details see our discussion of his paper *Über Modifikationen und exzeptionelle analytische Mengen*.

Grauert's elegant proof begins by implementing his now famous bumping technique where he constructs a (finite) increasing sequence $\{D_k\}$ of domains containing D such that at each step the restriction mapping $H^\nu(D_{j+1}, \mathcal{O}) \rightarrow$

$H^\nu(D_j, \mathcal{O})$ is surjective for all $\nu \geq 1$. The largest domain contains D as a relatively compact subset. Hence he proves that for D' sufficiently near D with $D \subset\subset D'$ the same surjectivity result holds. Actually in the final step of his proof he uses this for the sheaf of a holomorphic line bundle where the surjectivity is proved in exactly the same way. Having achieved the above

indicated surjectivity Grauert applies L. Schwartz' Fredholm theorem which states that if $\varphi : E \rightarrow F$ is a surjective, continuous linear map of Fréchet spaces and $\psi : E \rightarrow F$ is compact, then $\varphi + \psi$ has closed image of finite codimension. To apply this Grauert organizes a (finite) Leray covering \mathcal{U} of D' which is refined to a Leray covering \mathcal{V} of D and considers the mapping

$$\varphi : Z^q(\mathcal{U}, \mathcal{O}) \oplus C^{q-1}(\mathcal{V}, \mathcal{O}) \rightarrow Z^q(\mathcal{V}, \mathcal{O})$$

which is the sum of the restriction map R and the Čech boundary map δ . The cohomological surjectivity implies that this map is surjective. Hence, the Schwartz Theorem implies that the image of $\delta = \varphi - R$ has finite codimension. This is exactly the desired finiteness theorem.

Grauert uses this finite dimensionality to prove a result that is actually stronger than the holomorphic convexity of D : Given a boundary point x_0 , he enlarges D as above so that in addition D' contains a 1-codimensional complex submanifold S which contains x_0 but is entirely contained in the complement of D . Using the finite dimensionality of the cohomology of powers F^k of the line bundle defined by S as well as its restriction to S , for k sufficiently large he finds a section s of F^k which does not vanish at x_0 . Thus if h is the defining section of S in F , then $\frac{s}{h^k}$ is a meromorphic function on D' which is holomorphic on D with a pole at x_0 .

As the title of the article indicates, Grauert applies his theorem to prove that paracompact real analytic manifolds M can be embedded in Euclidean spaces of the expected dimension. For M compact this result was proved by Morrey using PDE methods slightly earlier. At the time Bruhat and Whitney had shown that M can be regarded as the set of real points of a complex manifold X , and Grauert then constructed a non-negative strictly plurisubharmonic function ρ on a neighborhood of M in X which vanishes exactly on M and otherwise has non-vanishing differential. If M is compact, then Grauert's solution to the Levi problem immediately implies that $T = \{\rho < \varepsilon\}$ is Stein and it is immediate that there is an everywhere maximal

rank, injective holomorphic map of T (shrunk a bit) to some \mathbb{C}^N . If M is not compact, more sophisticated arguments must be used. In particular, in order to prove that there is a Stein tube one must use a generalized version of the Behnke-Stein Runge theorem which is proved in [DG] (see our discussion of *Levisches Problem und Rungescher Satz*). Then Remmert's theorem for Stein manifolds yields the desired embedding.

In closing we should add that for weakly pseudoconvex domains in Stein manifolds, the Docquier-Grauert results are optimal. Strongly pseudoconvex domains in Stein spaces have been handled by Narasimhan [N1] with the analogous results to those discussed above. There are numerous partial results for weakly pseudoconvex domains, also in the case where singularities play a role. However, the general situation is far from being understood.

Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannifaltigkeiten

Here Grauert and Docquier begin by discussing nine conditions which are relevant for the study of the pseudoconvexity of a complex manifold. These are denoted by $(h, p_1, \dots, p_7, p_7^*)$ and are interrelated by a graph of implications with the condition h of holomorphic convexity being the strongest and p_7^* being the weakest. The latter condition is a weak version of the condition that a Hartogs figure cannot be mapped biholomorphically into the manifold so that its image is not relatively compact but the image of its Shilov boundary is compact. Riemann domains G which are unramified over a Stein manifold M are considered, and it is shown that if such a domain satisfies p_7^* , then it is Stein. It is therefore holomorphically convex, i.e., h is satisfied and consequently all of the conditions are fulfilled.

The basic idea of the proof is to apply Remmert's embedding theorem to embed M in some \mathbb{C}^n . Then, using the normal bundle of M in this embedding the domain G is thickened to an unramified domain \hat{G} over \mathbb{C}^n which also satisfies p_7^* . In that situation Oka's methods can be applied to \hat{G} to achieve the desired result.

Again using the idea of thickening a Remmert embedding, Runge approximation theorems are proved via Oka-Weil approximation for domains unramified over \mathbb{C}^n . A condition for a Stein domain M to be Runge in a complex manifold \tilde{M} (strongly simplified for our presentation) is that there is a continuous

increasing family of Stein domains M_t starting at M and ending at \check{M} . For $t_1 < t_2$ it is proved that M_{t_1} is Runge in M_{t_2} and then by the classical Runge Theorem of Behnke and Stein it follows that \check{M} is Stein.

This last mentioned result is really just a corollary of results proved in much greater generality. However, we wanted to particularly underline it, because it is exactly what is needed in [Gr6] for proving that Grauert's tube around a *non-compact* real analytic manifold is Stein.

Théorèmes de finitude pour la cohomologie des espaces complexes

Recall that a smooth function on, e.g., a domain in \mathbb{C}^n is strictly plurisubharmonic if its Levi form (complex Hessian) is positive-definite. Stein manifolds are those complex manifolds X which possess a strictly plurisubharmonic exhaustion. This is, in a certain sense, the solution of the Levi problem. In the work of Andreotti-Grauert, which we will now review, a q -Levi problem is solved. Just as the Levi problem for strongly pseudoconvex domains was handled by Narasimhan in the case of complex spaces, the context here is also for complex spaces. However, in order to explain the essential ideas, it is enough to consider the smooth case.

Let B be a bounded domain with smooth boundary ∂B in a complex manifold X . Andreotti and Grauert say that ∂B (or B) is q -pseudoconvex if the Levi form of a defining function has at least $n - q + 1$ positive eigenvalues, the case of $q = 0$ being reserved for compact manifolds. The main goal of the paper is to prove the finite dimensionality of $H^k(B, \mathcal{F})$ for $k \geq q$ where \mathcal{F} is a coherent sheaf on X . For example, if $\rho : X \rightarrow \mathbb{R}^{\geq 0}$ is an exhaustion which is q -pseudoconvex outside of a compact set K which is contained in a ρ -sublevel set B , then the restriction map $H^k(X, \mathcal{F}) \rightarrow H^k(B, \mathcal{F})$ is an isomorphism. Thus the finite dimensionality for X follows and if K is empty, then $H^k(X, \mathcal{F}) = 0$ for $k \geq q$.

A domain B is strictly q -pseudoconcave if ρ is a defining function as above and $-\rho$ is strictly q -pseudoconvex. With the addition of some technicalities in the case of singular spaces and coherent sheaves which are not locally free, the finiteness and vanishing theorems hold in the concave case for $0 \leq k \leq n - q$.

It is interesting that, in order to prove these global results, the main new work needed is of a local nature! This point actually comes up in Grauert's

previous paper, but since it is handled by classical methods, one tends to forget it. In that paper ∂B is strongly pseudoconvex. In a Stein coordinate chart U containing a given boundary point, Grauert constructs a bump on B to obtain a domain B_1 . It is of fundamental importance that the cohomology of $U \cap B_1$ vanishes, i.e., Cartan's Theorem B for this domain! Here, in the q -pseudoconvex case, the analogous "Lemma", along with Grauert's proof idea of bumping and then using the Fredholm Theorem of L. Schwartz, yields the proof. Of course there are substantial technical preparations which must be carried out.

Roughly speaking the above mentioned Lemma amounts to doing the following. In a local coordinate chart U at a boundary point ξ_0 one chooses a transversal polydisk of dimension $n - q + 1$ so that the restriction to it of the boundary defining function is strictly plurisubharmonic. Then one thickens it to obtain a holomorphic family of such polydisks parameterized by, e.g., a polydisk of complementary dimension. Then, as in the strongly pseudoconvex case, one creates a bumped region B_1 which intersects each transversal polydisk in a strongly pseudoconvex region and which only changes B in a compact region in U . It is now necessary to prove a cohomology vanishing theorem for $V := U \cap B$ for $k \geq q$, and the authors do exactly this by viewing V as a family of $(n - q + 1)$ -dimensional Stein domains. One of the main difficulties for this is proving the appropriate Runge theorem.

Algebraische Körper von automorphen Funktionen

Although the discussion here is in fact quite general, applying to any pseudoconcave complex space, the work in this paper is carried out in the special situation where modular and associated automorphic functions are playing an essential role. In fact only one example is considered, the quotient of the Siegel upper halfplane H by the modular group Γ , but as Borel later pointed out, there is a wide class of examples where the Andreotti-Grauert method applies.

A (connected) complex space X is said to be pseudoconcave if it contains a relatively compact open subset Y with the property that for every $x \in \partial Y$ there is a map $\varphi : \text{cl}(\Delta) \rightarrow \text{cl}(Y)$ which is holomorphic in a neighborhood of the closure of a 1-dimensional disk with image in the closure of Y with the properties that $\varphi(0) = x$ and $\varphi(\partial\Delta) \subset Y$. By thickening such

“disks” Andreotti and Grauert obtain a double covering of $\text{cl}(Y)$ by images of polydisks (one relatively compact in the other) so that their Shilov boundaries are contained in Y . Here the Shilov boundary of a polydisk $\Delta = \{|z_i| < 1, i = 1, \dots, n\}$ is the set where $|z_i| = 1$ for all i . They then apply Siegel’s method using the classical Schwarz Lemma to prove the following fact: The field $\mathbb{C}(X)$ of meromorphic functions on X is a finite algebraic extension $\mathbb{C}(f_1, \dots, f_k)[g]$ of the field of rational functions in k -algebraically independent meromorphic functions where $k \leq \dim(X)$. It should be mentioned that Andreotti went on to develop this theory in several ensuing works.

If, for example, X arises as the quotient \widehat{X}/Γ of some other space by the proper action of a discrete group, then the notion of pseudoconcavity can be formulated at the level of \widehat{X} . Andreotti and Grauert do this and then restrict their attention to the case where $\widehat{X} = H$ is the Siegel upper halfplane of complex $n \times n$ -matrices $Z = X + iY$ which are symmetric and where $Y > 0$. The discrete group which is of interest here is $\Gamma = \text{Sp}_{2n}(\mathbb{Z})$. It is acting properly and discontinuously so that the quotient H/Γ has the natural structure of a complex space. It is well known that Γ -periodic meromorphic functions, i.e., functions on the quotient, are important in more than one area of mathematics.

Using a well-known fundamental region Ω_0 for the Γ -action along with the strictly plurisubharmonic function $k(z) = -\log|Y|$, Andreotti and Grauert determine a region in H that descends to the quotient to show that it is pseudoconcave. An essential part of the proof is devoted to achieving the periodicity of $k(z)$ by minimizing it over Γ . This can be done, because the minima are taken on in the fundamental region.

As a consequence, the result on function fields can be applied in this case. This was known already, but was proved by using vastly more complicated methods. Furthermore the possibility of using pseudoconcavity in this area of mathematics was a totally new, extremely useful idea. It should be remarked that the same type of method can be used to prove that natural spaces of modular forms, e.g., for the canonical bundle, are finite dimensional. Furthermore, pseudoconcavity implies that the quotients H/Γ close up in projective embeddings to compact complex spaces to which all meromorphic functions extend.

Über Modifikationen und exzeptionelle analytische Mengen

Given a complex space X and a compact subvariety A one is interested in understanding when there is a complex space Y with a distinguished point $y \in Y$ and a surjective holomorphic mapping $\pi : X \rightarrow Y$ which is biholomorphic from $X \setminus A$ to $Y \setminus \{y\}$ and with $\pi(A) = \{y\}$. In other words, one would like to have sufficient conditions for A to be blown down to a point. In the projective algebraic setting in the case where X is a surface certain results were already known, e.g., for blowing down a smooth rational curve. In this beautiful paper Grauert answers this question in a general analytic setting in terms of the neighborhood geometry of A and its normal bundle. Underway he proves a number of results that can be considered as preparatory but which are also extremely useful in many areas of global complex geometry. As is often the case for Grauert, the guiding light is given by the notion of strong pseudoconvexity.

Here Grauert begins by noting that his solution to the Levi problem for relatively compact domains G in complex manifolds X had just been extended to the case where X is singular ([N1]). Using Remmert's reduction theorem, he observes that the result can be stated as follows: If G has strictly pseudoconvex boundary, then it contains a maximal compact analytic subset A which can be blown down to a finite number of points (corresponding to its connected components) by a map $\pi : G \rightarrow Y$ where Y is a Stein space. Conversely, if a connected compact analytic set can be blown down to a point, then it has a strongly pseudoconvex neighborhood. So it is natural to study the relation of this type of question to the pseudoconvexity of neighborhoods of the 0-section of the normal bundle of A or more generally for any bundle.

For line bundles F over a compact complex manifold X the importance of the notion of the positivity of a Hermitian bundle metric was known. One says that F is ample if some power F^k defines an embedding $X \hookrightarrow \mathbb{P}(\Gamma(X, F^k)^*)$ by mapping a point $x \in X$ to the hyperplane of sections which vanish at x . Kodaira's basic theorem states that F is ample if and only if it possesses a positive bundle metric. Since the region defined by $\|\cdot\| > 1$ can be regarded as a tubular neighborhood of the 0-section of F^* , Grauert reformulates positivity in terms of the strong pseudoconvexity of the 0-section of the dual bundle. It is important to emphasize that this also makes sense in the case where X is singular. He calls this property *schwach negativ*. Thus the embedding theorem can be stated as F is ample if and only if F^* is *schwach negativ*. This is then equivalent to the 0-section of F^* being the maximal compact

subset of F^* . It can be blown down to a Stein space which Grauert shows to be affine. Grauert's notion for vector bundles of higher rank is defined analogously: A vector bundle V over a compact complex space is said to be Grauert-positive if and only if the dual bundle V^* is schwach negativ in the above sense, i.e., its 0-section can be blown down. It should be remarked that in the vector bundle case the relation of Grauert's positivity condition to Griffiths-positivity is still not understood.

One of the main results of the paper is the embedding theorem: If a complex compact space X possesses vector bundle which is schwach negativ, then it is projective algebraic. The key is the Stein property for the blown down bundle space. Even in the case of line bundles $F \rightarrow X$ the result is new, because here singular spaces are allowed. This also gives a proof of the embedding theorem for Hodge spaces, another result of Kodaira in the smooth case.

The following is a more general version of the fact mentioned above, i.e., that the blown down dual bundle space is affine: Let F be the bundle of a divisor which has support A . Suppose $F|A$ is positive and that $X \setminus A$ contains no positive dimensional compact analytic subsets. Then $X \setminus A$ is affine and F is a positive bundle on X . The key ingredient for the proof, called a Hilfslemma by Grauert, is probably even more useful: A line bundle F over a compact complex space X is positive if and only if for every analytic subset A there exists $k > 0$ so that $F^k|A$ has a section which vanishes at some point of A but does not vanish identically.

Returning to the main theme of the paper, Grauert considers the notion of the normal bundle of a compact complex subvariety A of a complex space X . Due to the possible singular nature of these spaces, this must initially be regarded as a sheaf corresponding to the ideal sheaf \mathfrak{m} of A or more generally any coherent ideal sheaf \mathcal{I} which defines A . In typical Grauert fashion he is not phased by this difficulty but rather introduces the (quite natural) notion of a linear fiber space associated to a coherent sheaf. Locally over a trivializing neighborhood U this is a subvariety of $U \times \mathbb{C}^n$ where the fibers are subvector spaces of \mathbb{C}^n so that addition and scalar multiplication are well defined. Thus, given A and the ideal sheaf \mathcal{I} as above one has its normal linear fiber space $N_{\mathcal{I}}$ with its 0-section and the notion of schwach negativ has the obvious meaning. In elegant fashion Grauert transfers the pseudoconvexity of a neighborhood of the 0-section to that of a neighborhood

of A in X and proves the desired result: A can be blown down if for suitable \mathcal{I} the normal linear fiber space $N_{\mathcal{I}}$ is schwach negativ.

Of course the results in this paper have numerous applications. Even in the case of surfaces one needs Grauert's results to show that an irreducible curve C has negative self-intersection number if and only if it can be blown down to a point. Grauert's most general result in this direction is that a 1-dimensional subvariety in a surface can be blown down if and only if its self-intersection matrix is negative definite.

Direct image theorem

Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen

The proof of the Direct Image Theorem (*Bildgarbensatz*) ([Gr7]) is one of Hans Grauert's greatest accomplishments. We will state it here, say a bit about the proof and give an application mentioned by Grauert in the paper. As we wrote in ([H1]), the applications are so far reaching in complex analytic geometry that it would be unimaginable to work in the area without having it available.

Let us turn to the setting of complex analysis at the time (the late 1950's). A great deal was known, at least compared to ten years before. The notion of a complex space had been clarified, Grauert already had a huge experience as *Handarbeiter* in dealing with problems of cohomology, e.g., the Oka Theorems and his proof of Theorems A and B were behind him, and he understood very well how to deal with refining covers and using the relevant Fréchet spaces and compact operators between them. Given all of this he was in a position to consider the problem of the coherence of direct images of coherent sheaves.

The initial geometric context of this theorem is quite simple. One begins with a holomorphic map $F : X \rightarrow Y$ between complex spaces which is defined in the most naive way, e.g., locally it is given by holomorphic functions. Associated to an open subset U in Y one has the algebra $\mathcal{O}_X(F^{-1}(U))$ on its preimage. This has the structure of an $\mathcal{O}_Y(U)$ -module which is given by multiplication by lifted functions $F^*(f)$. This presheaf defines a sheaf $F_*(\mathcal{O}_X)$ on Y . It contains a great deal of information about the map F ,

in particular about its singularities. It would clearly be of interest to know whether or not it is coherent.

If indeed the direct image $F_*(\mathcal{O}_X)$ is coherent, then its support is a closed analytic subset of Y . Hence, the correct condition for the direct image theorem to hold must be something that guarantees that images of analytic subsets are analytic. At the time, Remmert's theorem, which guarantees that this is the case for F being a proper (holomorphic) map, had been proved. It should be underlined that even the notion *proper*, i.e., inverse images of compact sets are compact, was rather new. Cartan had introduced this in the 1930's while discussing the fact that the action of group of automorphisms on a bounded domain is proper and the notion was explicitly described in Bourbaki. Kuhlmann had pointed out that there is a weaker notion (semi-proper) and had proved that Remmert's theorem holds for this kind of map. Stein and Grauert were always interested in understanding holomorphic equivalence relations and finding a good condition which would insure that the quotient is analytic. In fact, one of Grauert's last papers was devoted to a situation where a sort of semi-properness was built into the assumptions.

In any case, at the time when Grauert considered the problem of the coherence of direct images of coherent sheaves, much was known, but even at the set-theoretic level (Remmert's theorem) things had not settled in. There were also a huge number of foundational issues. For one, even the notion of an analytic morphism had to be improved. One reason for this, at least from Grauert's point of view, was that the entire project had to be carried out in the context of complex spaces where the structure sheaf is allowed to have *nilpotent* elements. This means that the local model is as before an analytic set A in some domain D in \mathbb{C}^n , but the sheaf of germs of holomorphic functions is $\mathcal{O}_D/\mathcal{I}_A$ where \mathcal{I}_A is any coherent ideal sheaf which defines A as its 0-set. Since the structure sheaf is not necessarily a subsheaf of the sheaf of continuous functions, the classical definition of a map being holomorphic, i.e., pullbacks of holomorphic germs are required to be holomorphic, is not sufficient. A holomorphic map is then a pair (F_0, F_1) where $F_0 : X \rightarrow Y$ is a usual map of sets and F_1 is a map of structure that encodes the notion of pullback, a continuous homomorphism of sheaves of algebras $F_1 : Y \times_{F_0} \mathcal{O}_Y \rightarrow \mathcal{O}_X$. Grauert begins his paper with a rather long discourse on how to deal with these *new* complex spaces where he proved the key theorems such as Theorems A and B in this more general setting.

Given a morphism $F : X \rightarrow Y$ and sheaf \mathcal{S} of \mathcal{O}_X -modules, F_1 is applied to equip the direct image sheaf $\pi_*(\mathcal{S})$ with the structure of a sheaf of \mathcal{O}_Y -modules. One can go an important step further: For U open in Y and every $q \geq 0$ the cohomology space $H^q(F^{-1}(U), \mathcal{S})$ is equipped by means of F_1 with the structure of a $\mathcal{O}_X(U)$ -module. Hence, for every q we have direct image sheaf $R^q F_*(\mathcal{S})$. The following is then the *Bildgarbensatz*.

Theorem. *If $F : X \rightarrow Y$ is a proper holomorphic map of complex spaces and \mathcal{S} is a coherent sheaf of \mathcal{O}_X -modules, then for every $q \geq 0$ the direct image sheaf $R^q F_*(\mathcal{S})$ is coherent.*

There were germs of this result around at the time, e.g., Remmert had proved a result in special case of finite maps, Grauert and Remmert had proved it in the situation where $X = Y \times \mathbb{P}_n$ and F is the obvious projection and Grothendieck had proved the far simpler algebraic version. However, this result, along with the Oka Principle papers, brought complex analysis into a new era. Only five years before members of the Behnke seminar were trying to understand \sqrt{xy} !

Let us quote Grauert ([G10], p. 446) when discussing the main difficulties in the proof which involve a power series argument to handle the directions transversal to a fixed fiber X_0 . *Roughly speaking, the proof of the direct image theorem uses power series expansion whose coefficients are cohomology classes on one fixed fiber. The coefficients are obtained recurrently and with estimates. Cohomology spaces carry natural Fréchet space structures. However, to get a convergent power series by recurrent formula with estimates, one needs a fixed norm for the iteration process instead of the infinite sequence of semi-norms. One key point of the proof of the direct image is to replace a cocycle with estimates for a weak norm by another cocycle with estimate for a stronger norm modulo a coboundary with estimates for an even weaker norm.*

Grauert began this article with a rather lengthy discussion of the importance of the cohomology of a certain direct image sheaf for the study of moduli spaces M . In that case $F : X \rightarrow M$ is a usual (surjective) proper holomorphic map of complex manifolds which is everywhere of maximal rank. The relevant sheaf is Θ_X , the sheaf of germs of holomorphic vector fields on X . For example, if X_y denotes the fiber over $y \in M$ and $r_q(y)$ is the dimension of $H^q(X_y, \Theta_y)$, then the semicontinuity of $r_q(y)$, which was proved by Kodaira

and Spencer using the method of harmonic integrals, follows immediately from the direct image theorem. In fact, in much greater generality the direct image theorem implies the semicontinuity for any coherent sheaf provided the proper map $F : X \rightarrow Y$ is flat.

It took the complex analysis community a number of years to understand and somewhat simplify Grauert's proof (see, e.g., [FK] and [N2]). According to Grauert, the simplest proof can now be found in [GR2].

Akademischer Lehrer

Let me close this note with some personal comments. I was introduced to the “German school” of complex analysis in the late 1960's in the Stanford lectures of Aldo Andreotti, where I was the only student. In the first semester of these lectures Andreotti explained a number of Grauert's results (some of them discussed above) in a beautiful way. Although I had been a student for a while, this was the first time that I felt that I had seen “the truth”. Of course Andreotti was a master lecturer, but the truth, I sensed, was embedded in Grauert's work. A few months later, when I realized I should prepare for my German language exam, luck struck again: My advisor, Halsey Royden, had explained in seminars some of his ideas on metrics on Teichmüller space. As a result I optimistically thought it might be good to go back to the basics and read Hermann Weyl's book *Die Idee der Riemannschen Fläche* and then jump to the modern developments and study Grauert's paper *Über Modifikationen und exzeptionelle analytische Mengen*. Up to this point I had had very little experience reading mathematics and assumed this is the way it should be! Looking back, it is hard for me to believe that I had been so naively audacious! A year or so later during my postdoc time in Pisa, I told my friends that these two works were the only things I really understood. At the time I was embarrassed to admit this but soon realized my good fortune!

In the second semester of the above-mentioned course Andreotti “asked” me to lecture on various topics involving $\bar{\partial}$ at the boundary, Hans Levy's extension, etc.. The audience consisted of Andreotti and Wilhelm Stoll. This began my lasting friendship with Stoll. He came from another “Schwerpunkt” of complex analysis, namely from the Tübingen group of Hellmuth Kneser. The complex analysis of Tübingen was, to a certain extent, related to that of the Münster school, e.g., they had competing theories of meromorphic

maps. However, Kneser and Stoll went in other directions, proving continuation theorems under assumptions of bounded volume, and then building the foundations of value distribution theory in several complex variables.

My close relationship with Stoll continued during the almost 10 years I spent at Notre Dame where, particularly due to Stoll's connections, the faculty had close ties to German mathematicians. In my very first year there Stein visited for a semester. His energy, openness and obvious love of mathematics made a great impression on me. The next year Remmert came, and it was a great honor for me to drive him around town as he reviewed periods in which he had also been a guest at Notre Dame. In those days it was a bit non-standard to go from the US to Oberwolfach for a week, but when he invited me I jumped at the opportunity. In my Oberwolfach lecture Grauert, Remmert, Stein and Forster were in the first row. I figured if I could get through that I could get through anything!

Grauert was a kind, warm person of very few words. Sometimes he looked formal, but he was not. In the above-mentioned conference Douady lectured on his construction of the versal deformation of a complex space. Grauert, who had developed his own version of this theory, sat in the first row. Douady, who was dressed in a silk-like Hawaiian shirt which was not completely buttoned and was rather dirty because the night before he had slept in the forest, explained his puzzles and made silly jokes. Grauert remained quiet and respectful, only caring about the content. Later on at a memorial meeting in honor of Andreotti, who had passed away at a very early age, one could see that Grauert and Douady were very close.

Grauert didn't say much, but when he did he meant it! A comment of "good, continue on" to a young speaker after a talk really meant something. Similarly, his way of praising a student was often "Das können wir so machen". He was a key referee for one of our research concentrations sponsored by the DFG. As a 50-year-old I nervously appeared in front of Grauert for his comments: "almost everything is good, but the mathematics in this subproject is not important". Of course I dropped the subproject from the proposal. One might think of Grauert as being opinionated, but his opinions were based on serious thought; anything he said or wrote should be taken seriously!

Hans Grauert was an "Akademischer Lehrer" in the sense of Humboldt. He didn't teach a subject because it was in the syllabus; he taught it because

he had thought about it and knew it was important. He had a large number of doctoral students, more than 40, and he was proud of it. I know that in every case he had thought through their projects and made notes on the little pieces of paper that he carried around. One of our colleagues who was a student of Grauert remembers seeing the same piece of paper every time he came to Grauert's office hours.

Even though he was not the type of person to be heavily involved with the global organization of science, he did his duty, e.g., as managing editor of *Mathematische Annalen* and working in various capacities with the DFG. He certainly respected the traditions of Göttingen and was proud to have been President of the Göttingen Academy of Sciences. His strong points were, however, in the classroom where he focused on important phenomena in mathematics, in his one-on-one work with his students and of course in his remarkable research.

Those of us who have had the privilege of knowing Hans Grauert will not forget him. Fortunately, his deep ideas have survived him in his written works. Let us hope that his high standards of excellence in every aspect of our science will be carried on by future generations.

References

- [AMS] A tribute to Hans Grauert, ed. A. Huckleberry and T. Peternell, Notices of the AMS, (to appear 2013)
- [AG1] Andreotti, A. and Grauert, H.: Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France **90**, 193-259 (1962)
- [AG2] Andreotti, A. and Grauert, H.: Algebraische Körper von automorphen Funktionen, Nachr. Akad. Wiss. Göttingen, II. Math-Phys. **K1.3**, 39-48 (1961)
- [C] Cartan, H.: Espace fibrés analytique, Symposium International de Topologia Algebraica, Mexico, 97-121 (1958)
- [DG] Docquier, F. and Grauert, H.: Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannifaltigkeiten, Math. Ann **140**, 94-123 (1960)

- [FK] Forster, O. and Knorr, K.: Ein Beweis des Grauert'schen Bildgarbensatzes nach Ideen von B. Malgrange, *Manuscripta Math.* **5**, 19-44 (1971)
- [F] Forstnerič, F.: Stein manifolds and holomorphic mappings, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3.Folge, Vol. 56, Springer Verlag, 2011
- [G] Gromov, M.: Oka's principle for holomorphic sections of elliptic bundles, *J. Amer. Math. Soc.*, **2**, 851-897 (1989)
- [Gr1] Grauert, H.: Charakterisierung der holomorph vollständigen komplexen Räume, *Math. Ann.* **129**, 233-259 (1955)
- [Gr2] Grauert, H.: Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik, *Math. Ann.* **131**, 38-75 (1956)
- [Gr3] Grauert, H.: Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen, *Math. Ann.* **133**, 139-159 (1957)
- [Gr4] Grauert, H.: Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen, *Math. Ann.* **133**, 450-472 (1957)
- [Gr5] Grauert, H.: Analytische Faserungen über holomorph-vollständigen Räumen, *Math. Ann.* **135**, 263-273 (1958)
- [Gr6] Grauert, H.: On Levi's Problem and the Imbedding of Real-Analytic manifolds, *Annals of Math. II Ser.* **68**, 460-472 (1958)
- [Gr7] Grauert, H.: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, *Publ. Math. Paris IHES* **5**, 233-292 (1960)
- [Gr8] Grauert, H.: Über Modifikationen und exzeptionelle analytische Mengen, *Math. Ann.* **146**, 331-368 (1962)
- [Gr9] Grauert, H.: Bemerkenswerte pseudokonvexe Mannigfaltigkeiten, *Math. Z.* **51** 377-391 (1963)
- [G10] Grauert, H.: Selected works with commentaries, I. and II., Springer-Verlag (1994)

- [GR1] Grauert, H. and Remmert, R.: Komplexe Räume, Math. Ann. **136**, 245-318 (1958)
- [GR2] Grauert, H. and Remmert, R.: Coherent analytic sheaves, Grundlehren der mathematischen Wissenschaften **265** Springer-Verlag (1984)
- [H1] Huckleberry, A.: Hans Grauert: Mathematiker pur, Mitt. Dtsch. Math. Verein. **16**, no. 2, 75-77 (2008)
- [H2] Huckleberry, A.: Hans Grauert: Mathematiker pur, Notices of the AMS **56**, no. 1, 38-41 (2009)
- [H3] Huckleberry, A.: Karl Stein (1913-2000), Jahresber. Deutsch. Math.-Verein **110**, no. 4, 195-206 (2008)
- [HP] Hulek, K. and Peternell, T.: Henri Cartan, ein französischer Freund, Jahresber. Dtsch. Math.-Ver. 111.Band, Heft 2, 85-94 (2009)
- [N1] Narasimhan, R.: The Levi problem on complex spaces, Math. Ann. **142** (1960) 355-365, Part II. in Math. Ann. **146** 195-216 (1962)
- [N2] Narasimham, R.: Grauert's theorem on direct images of coherent sheaves, Séminaire de Mathématique Supérieures, No.40 (Été 1969), Les Presses de l'Université de Montréal, Montreal, Que. 79 pp.
- [O] Ohsawa, T.: On complete Kähler domains with C^1 -boundary, Pub. RIMS Kyoto **16**, 929-940 (1980)
- [R] Remmert, R.: Complex Analysis in "Sturm und Drang", Mathematical Intelligencer, vol. 17, no. 2, Springer Verlag (1995)
- [S] Stein, K.: Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem, Math. Ann. **123**, 201-222 (1951)

Alan Huckleberry
 Ruhr Universität Bochum
 and
 Jacobs University in Bremen
 ahuck@cplx.rub.de